# OTFS Modulation: A Zak Transform Perspective

**Akshay S. Bondre and Christ D. Richmond**

**University Center of Excellence**

**November 8th 2021, Monday, 12pm EDT**

*Signals, Information, Inference, & Learning (SIIL) Group*

Arizona State University

# Outline

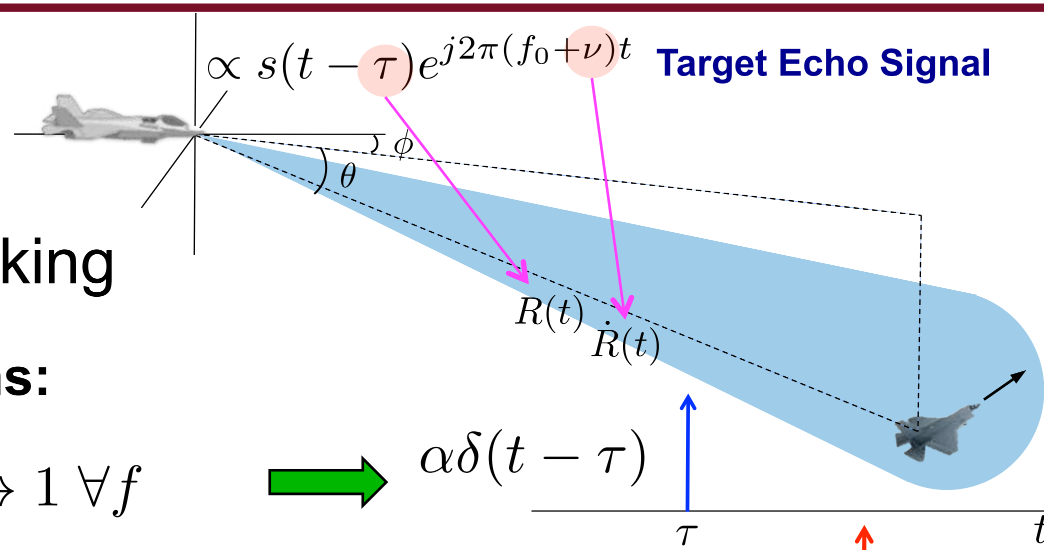
- **Introduction**
  - **Radar waveforms and signal processing**
- Akshay discussion
- Closing remarks

# Radar* and Waveforms

**Radar Goals:**

Detection,

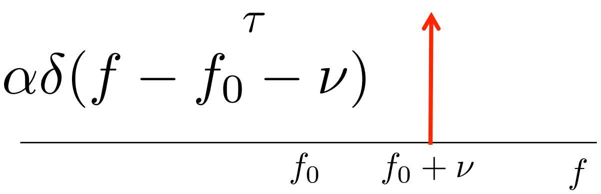
localization, and tracking




- **Ideal Radar Waveforms:**

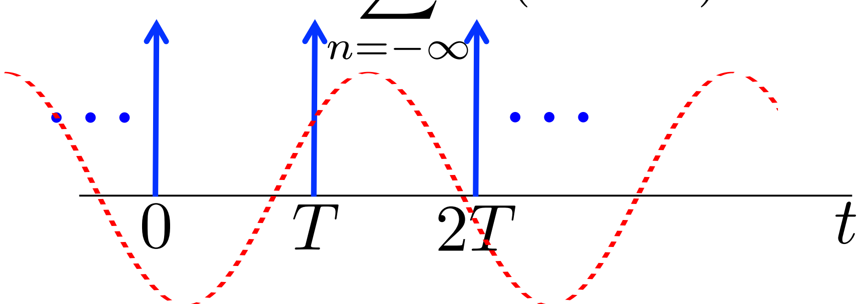
**Delay** $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 \; \forall f$

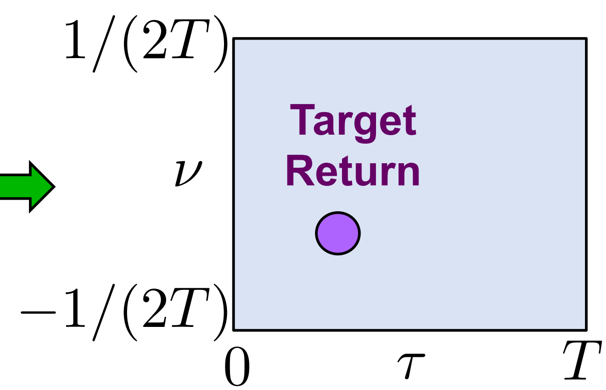
**Doppler** $e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f-f_0)$




- **Ideal Pulsed-Doppler Radar Waveform:**

$$e^{j2\pi f_0 t} \sum_{n=-\infty}^{\infty} \delta(t-nT)$$



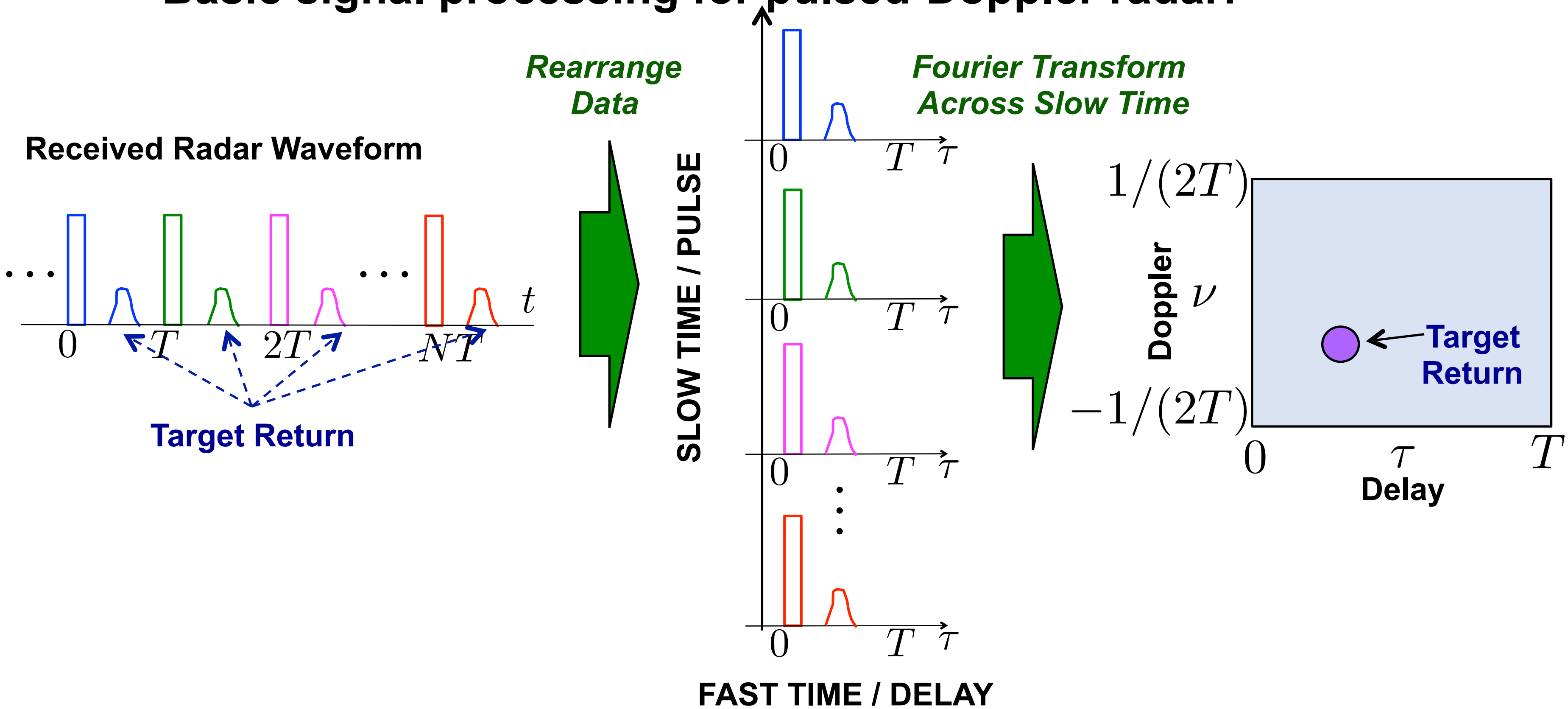

***Simultaneous Resolution in Delay and Doppler***



*RADAR = RAdio Detection And Ranging

# Radar Signal Processing

- **Basic signal processing for pulsed-Doppler radar:**




- **Zak Transform (ZT) formalizes this basic radar processing**
- **ZT and OTFS modulation use ideal pulsed-Doppler waveforms as fundamental basis expansion functions**

# Outline

- Introduction
- **Akshay discussion**
- Closing remarks

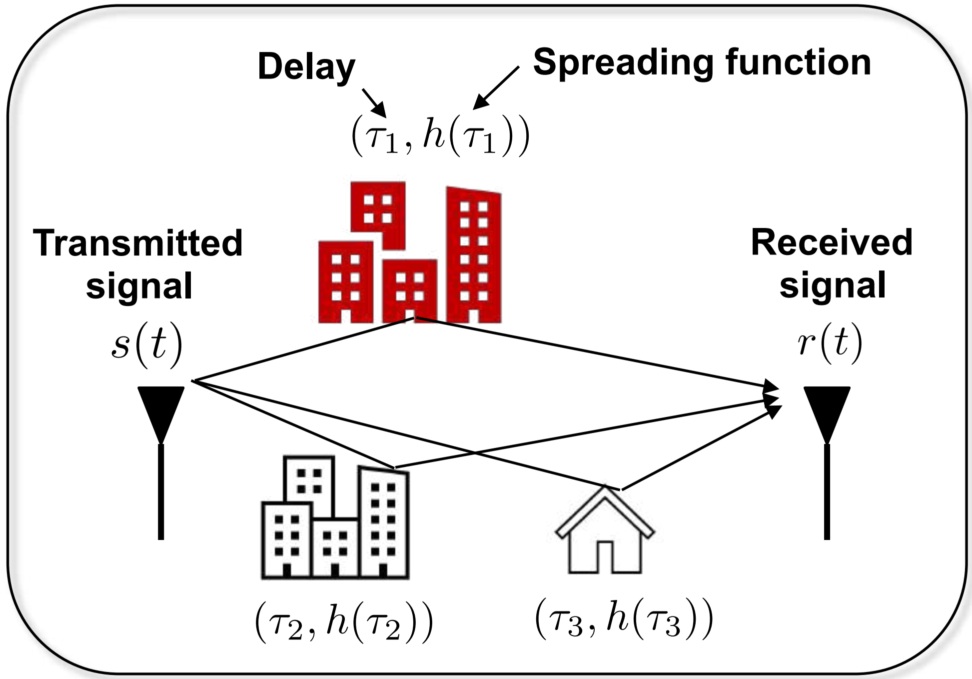
# Outline

- **Wireless Communications Channel Models**
- Orthogonal Frequency Division Multiplexing (OFDM)
- The Zak Transform
- OTFS: Modulation based on the Zak Transform
- Summary

# Delay-Spread Channels

- **Stationary scatterers introducing different delays**
- **Spreading function** $h(\tau)$
- **Received signal**

$$r(t) = h(\tau_1)s(t-\tau_1) + h(\tau_2)s(t-\tau_2) + h(\tau_3)s(t-\tau_3)$$




- **In general**

$$r(t) = \int_0^\infty h(\tau)s(t-\tau)d\tau$$

→ **LTI system with impulse response** $h(\tau)$

# Doubly-Spread Channels

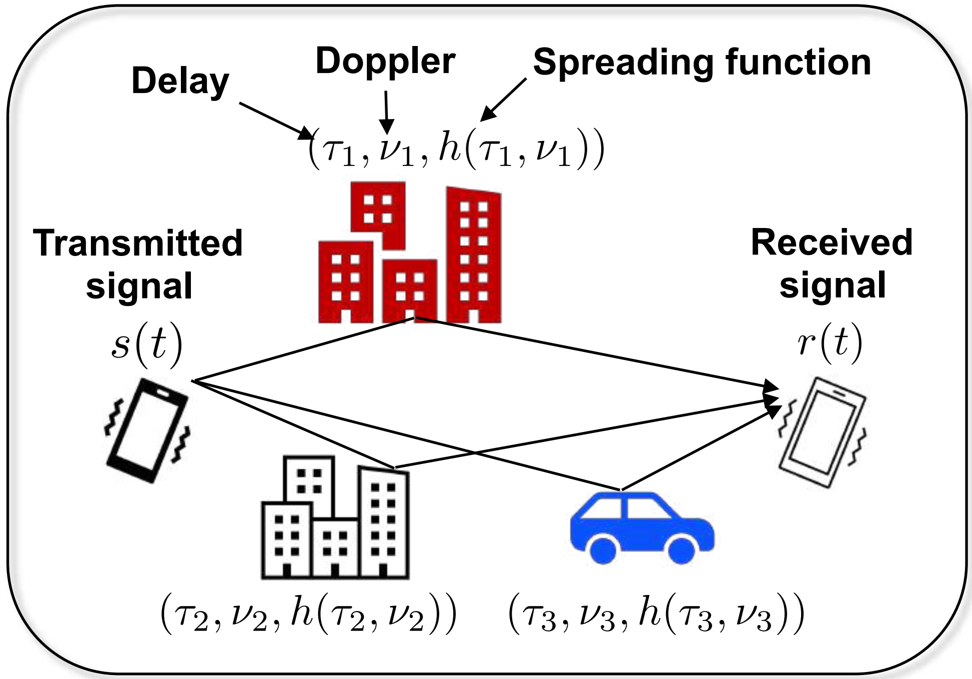
- **Moving scatterers introducing different Doppler shifts**

- **Spreading function** $h(\tau, \nu)$

$$r(t) = h(\tau_1, \nu_1)e^{j2\pi\nu_1(t-\tau_1)}s(t-\tau_1) + h(\tau_2, \nu_2)e^{j2\pi\nu_2(t-\tau_2)}s(t-\tau_2) + h(\tau_3, \nu_3)e^{j2\pi\nu_1(t-\tau_3)}s(t-\tau_3)$$

- **Received signal**

$$r(t) = \iint h(\tau, \nu)e^{j2\pi\nu(t-\tau)}s(t-\tau)d\tau d\nu$$



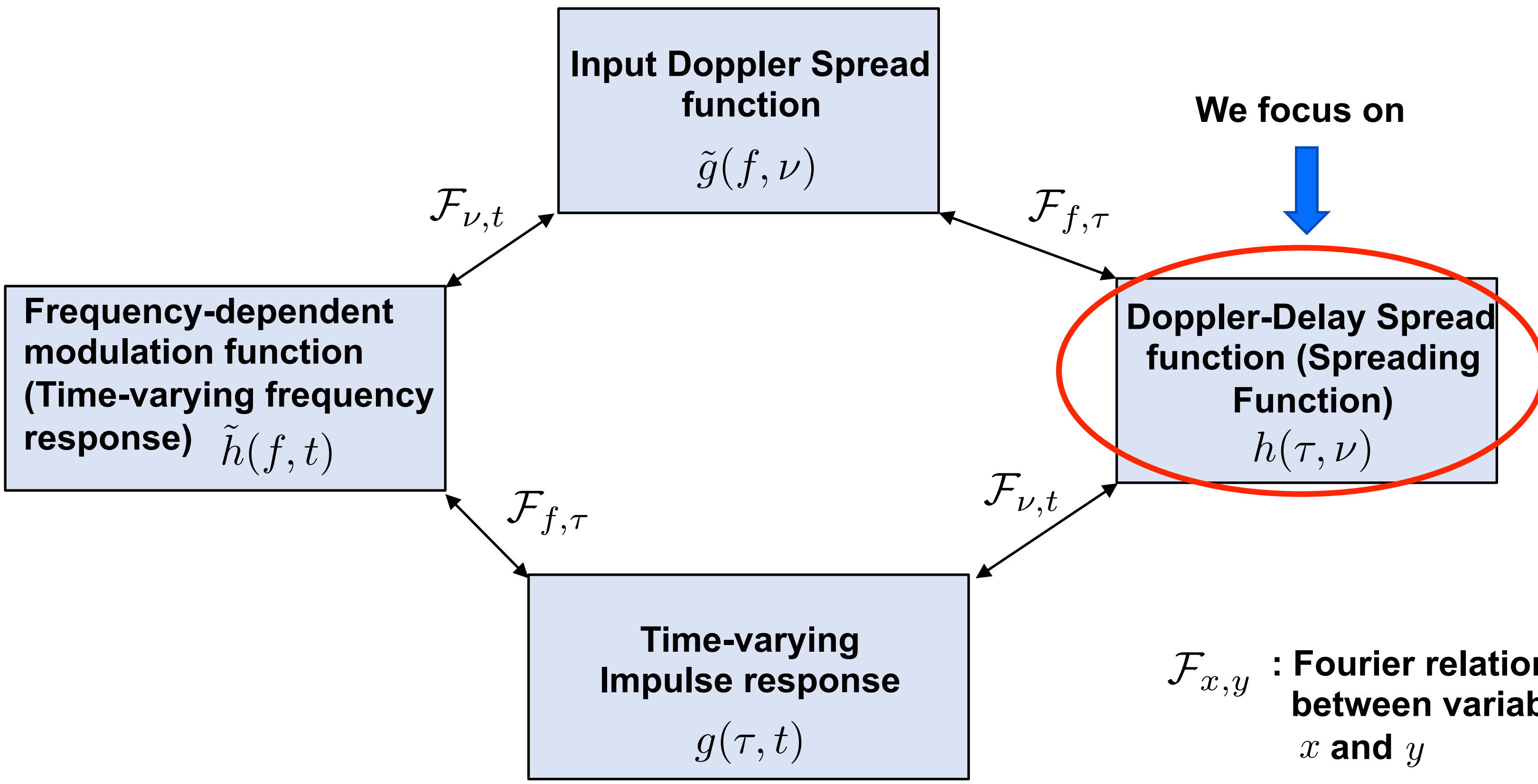

- **Linear Time-Varying (LTV) System with spreading function** $h(\tau, \nu)$



*P. Bello, "Characterization of randomly time-variant linear channels", IEEE Trans. on Comm. Syst., vol. 11, no. 4, pp. 360–393, December 1963.

# LTV Channel Representation

- **Linear time-varying (LTV) channels can be represented using the following functions*:**




$\mathcal{F}_{x,y}$ **: Fourier relationship between variables $x$ and $y$**

*P. Bello, "Characterization of randomly time-variant linear channels", IEEE Trans. on Comm. Syst., vol. 11, no. 4, pp. 360–393, December 1963.

# Outline

- Wireless Communications Channel Models
- **Orthogonal Frequency Division Multiplexing (OFDM)**
- The Zak Transform
- OTFS: Modulation based on the Zak Transform
- Summary

# Orthogonal Frequency Division Multiplexing (OFDM) Review

- **Discrete-time (DT) LTI channel with impulse response $h[k]$**

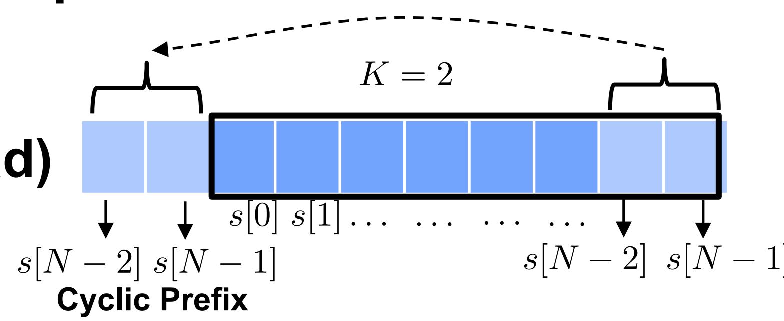
$$S[m] \overset{\text{IDFT}}{\longleftrightarrow} s[n] \Rightarrow \boxed{h[k]} \Rightarrow r[n] = \sum_{k=0}^{K-1} h[k]s[n-k]$$

$m, n = 0, \dots, N-1$

**LTI Channel (Delay Spread)**

$$\overset{\text{DFT}}{\longleftrightarrow} \boxed{R[m] = H[m]S[m]}$$

- $S[m], m = 0, \dots, N-1$ ➡ **QAM symbols placed in Fourier domain**




- **Cyclic prefix of length $K$ (delay spread)**
  - **Removes inter-block interference**

- **Inter-carrier interference in presence of significant Doppler spreads for LTV channels**

- **OFDM ➡ Modulation based on DFT multiplication for LTI channels**




A. Goldsmith, "Wireless communications", *Cambridge University Press*, 2005.

# Outline

- Wireless Communications Channel Models
- Orthogonal Frequency Division Multiplexing (OFDM)
- **The Zak Transform**
- OTFS: Modulation based on the Zak Transform
- Summary

# The Zak Transform

- **Fourier transform** ➡ $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$ **(Frequency domain)**
- **Zak Transform** ➡ $x(t) \overset{\mathcal{Z}}{\longleftrightarrow} \mathcal{Z}_x(\tau, \nu)$ **(Delay-Doppler domain)**

$$\mathcal{Z}_x(\tau,\nu) = \sqrt{T} \sum_{n=-\infty}^{\infty} x(\tau + nT) e^{-j2\pi n \nu T}$$

**For a fixed** $\tau$ ➡ $x(\tau + nT) = x_\tau[n]$

**Collection of DTFTs** ➡ $= \sqrt{T} \sum_{n=-\infty}^{\infty} x_\tau[n] e^{-j2\pi n\nu T}$

  - **Parameter** $T > 0$
  - **Periodic in** $\nu$ **with period** $F = 1/T$
  - **Quasi-periodic in** $\tau$ **with period** $T$

- **Fourier basis: Complex Exponentials** $e^{j2\pi f t}$ **,** $f \in \mathbb{R}$ $\quad x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} dt$
- **Zak basis: Doppler shifted and delayed impulse trains** $p_{\tau,\nu}(t)$

$p_{0,0}(t) = \sqrt{T}\sum_n \delta(t - nT)$ ⟶ **Doppler** $\nu$ ⟶ $e^{j2\pi\nu t} p_{0,0}(t)$ ⟶ **Delay** $\tau$ ⟶ $p_{\tau,\nu}(t) = e^{j2\pi\nu(t-\tau)} p_{0,0}(t-\tau)$

$$p_{\tau,\nu}(t) = \sqrt{T}\sum_n \delta(t - \tau - nT) e^{j2\pi\nu n T}$$

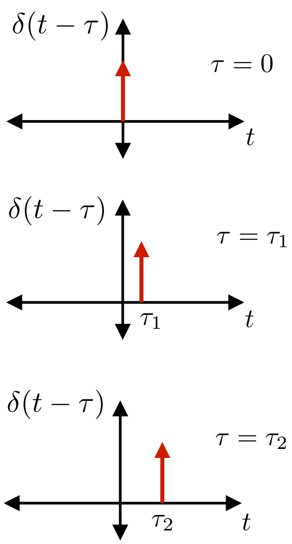


S.K. Mohammed, "Derivation of OTFS Modulation From First Principles", *IEEE Trans. Veh. Tech.*, Aug 2021.

# Zak Basis Functions

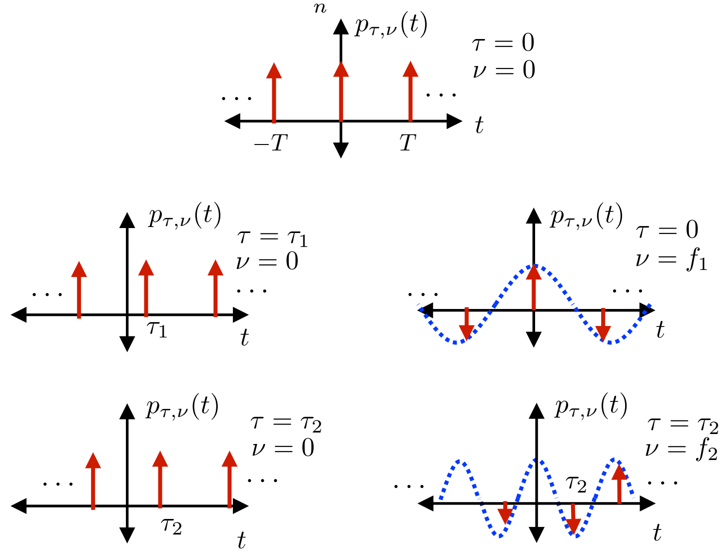
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

**Delayed impulse basis $\delta(t-\tau)$**
**Parameter: Delay $\tau$**



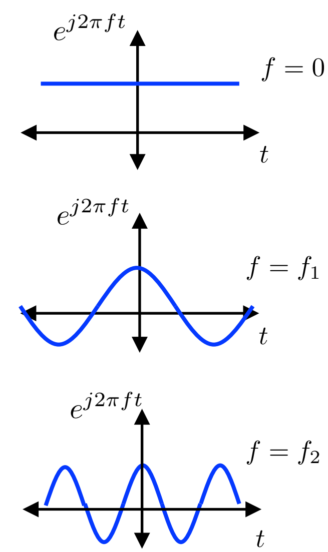

**Zak Basis, Parameters: Delay $\tau$ and Doppler $\nu$**
**(Delayed and Doppler shifted impulses)**

$$p_{\tau,\nu}(t) = \sqrt{T}\sum_{n} \delta(t-\tau-nT)e^{j2\pi\nu nT}$$




$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

**Complex Exponential Basis $e^{j2\pi ft}$**
**Parameter: Frequency $f$**




**Zak Analysis and Synthesis Equations**

$$\mathcal{Z}_x(\tau,\nu) = \int_{-\infty}^{\infty} x(t)p^*_{\tau,\nu}(t)dt$$

$$x(t) = \int_{0}^{T}\int_{-F/2}^{F/2} \mathcal{Z}_x(\tau,\nu)p_{\tau,\nu}(t)d\tau d\nu$$



S.K. Mohammed, "Derivation of OTFS Modulation From First Principles", *IEEE Trans. Veh. Tech.*, Aug 2021.

# Comparison with the Fourier Transform

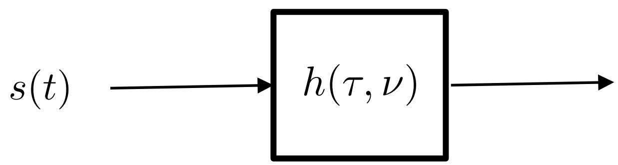
| | Fourier Transform | Zak Transform |
|---|---|---|
| Domain | Frequency $f$ | Delay-Doppler $(\tau, \nu)$ |
| Basis functions | $e^{j2\pi ft}$ | $p_{\tau,\nu}(t)$ |
| Analysis Equation | $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$ | $\mathcal{Z}_x(\tau,\nu) = \int_{-\infty}^{\infty} x(t)p^*_{\tau,\nu}(t)dt$ |
| Synthesis Equation | $x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}dt$ | $x(t) = \int_{0}^{T}\int_{-F/2}^{F/2} \mathcal{Z}_x(\tau,\nu)p_{\tau,\nu}(t)d\tau d\nu$ |

# Outline

- Wireless Communications Channel Models
- Orthogonal Frequency Division Multiplexing (OFDM)
- The Zak Transform
- **OTFS: Modulation based on the Zak Transform**
- Summary

# Underspread LTV Channels

- **Doppler-delay spreading function** $h(\tau,\nu)$




**LTV Channel**

$$r(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau,\nu)e^{j2\pi\nu(t-\tau)}s(t-\tau)d\tau d\nu$$

- **Wireless channels are generally underspread* such that**
  - $h(\tau,\nu)$ **with compact support** $[0,\tau_m]\times[-\frac{\nu_m}{2},\frac{\nu_m}{2}]$ **such that** $\tau_m \cdot \nu_m < 1$

- **A number** $T_m$ **exists such that** $T_m > \tau_m$ **and** $1/T_m = F_m > \nu_m$

**Output signal** ➡ $r(t) = \int_{0}^{T_m}\int_{-F_m/2}^{F_m/2} h(\tau,\nu)e^{j2\pi\nu(t-\tau)}s(t-\tau)d\tau d\nu$

*W. Kozek, "On the transfer function calculus for underspread LTV channels," *IEEE Trans. Sig. Proc.*, Jan. 1997.

# Zak Transform and Underspread LTV Channels

$$r(t) = \int_0^{T_m} \int_{-F_m/2}^{F_m/2} h(\tau,\nu) e^{j2\pi\nu(t-\tau)} s(t-\tau) d\tau d\nu$$

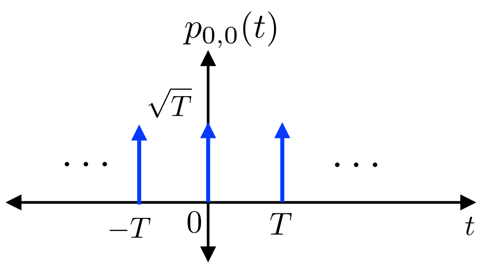
**Zak synthesis equation**

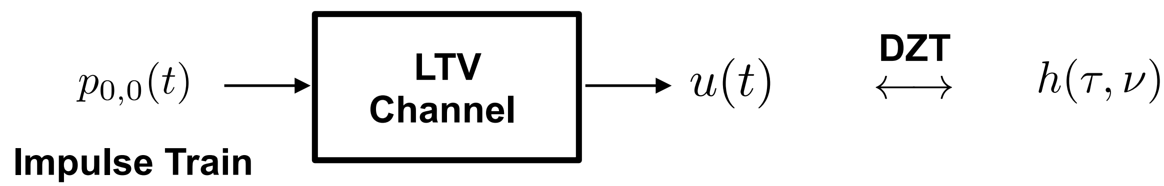
$$x(t) = \int_0^{T} \int_{-F/2}^{F/2} \mathcal{Z}_x(\tau,\nu) p_{\tau,\nu}(t) d\tau d\nu$$

- **Consider the input signal as** $s(t) = p_{0,0}(t) = \sqrt{T} \sum_n \delta(t - nT)$

$$u(t) = \int_0^{T_m} \int_{-F_m/2}^{F_m/2} h(\tau,\nu) \underbrace{e^{j2\pi\nu(t-\tau)} p_{0,0}(t-\tau)}_{p_{\tau,\nu}(t)} d\tau d\nu = \int_0^{T_m} \int_{-F_m/2}^{F_m/2} h(\tau,\nu) p_{\tau,\nu}(t) d\tau d\nu$$

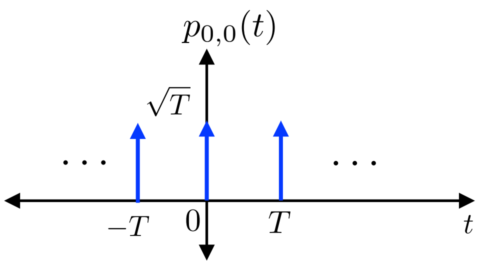
$$u(t) \overset{\mathcal{Z}}{\longleftrightarrow} \mathcal{Z}_u(\tau,\nu) = h(\tau,\nu)$$




***The Doppler-delay spread function*** $h(\tau,\nu)$ ***is the Zak transform of the output signal when the input is the impulse train*** $p_{0,0}(t)$

# Zak Domain LTV Channel Representation




$p_{0,0}(t)$ → **LTV Channel** → $u(t)$ $\overset{\text{DZT}}{\longleftrightarrow}$ $h(\tau, \nu)$

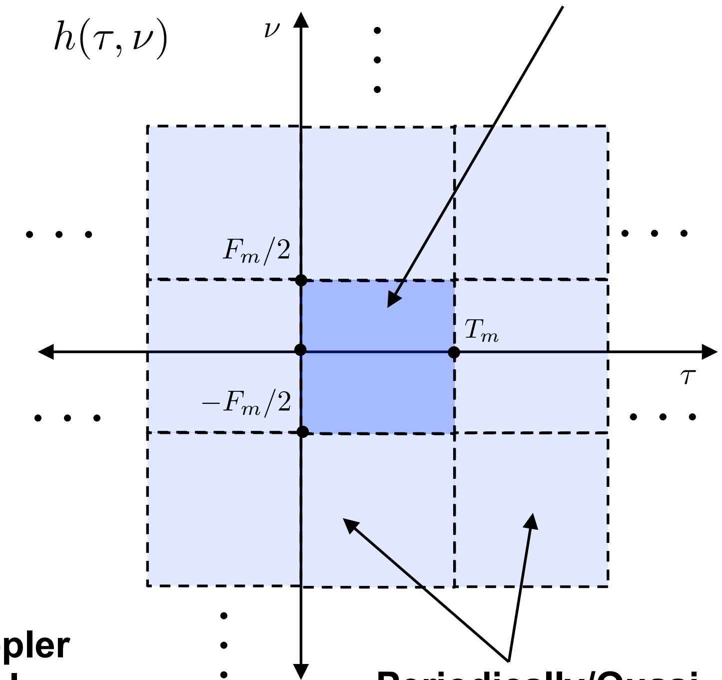
**Impulse Train** | **Impulse Train Response** | **Delay-Doppler (Zak) response**

**Underspread LTV Systems**

**LTI Systems**

**Impulse response** $h(t)$ $\overset{\mathcal{F}}{\longleftrightarrow}$ **Frequency response** $H(f)$

**Impulse train response** $u(t)$ $\overset{\mathcal{Z}}{\longleftrightarrow}$ **Delay-Doppler response** $h(\tau, \nu)$




**Periodically/Quasi-Periodically repeating copies**

**We can interpret the spread function $h(\tau, \nu)$ as the delay-Doppler response (or Zak response) of the underspread LTV channel.**

# Zak Domain Channel Equation for Underspread LTV Channels

- **Time domain equation** $r(t) = \int_0^{T_m} \int_{-F_m/2}^{F_m/2} h(\tau,\nu) e^{j2\pi\nu(t-\tau)} s(t-\tau) d\tau d\nu$

- **Zak domain channel equation**

$$\mathcal{Z}_r(\tau,\nu) = \sqrt{T} \sum_{n=-\infty}^{\infty} r(\tau + nT) e^{-j2\pi n\nu T}$$

$$= \sqrt{T} \sum_{n=-\infty}^{\infty} \int_0^{T_m} \int_{-F_m/2}^{F_m/2} h(\tau',\nu') e^{j2\pi\nu'(\tau+nT-\tau')} s(\tau + nT - \tau') d\tau' d\nu' e^{-j2\pi n\nu T}$$

$$= \int_0^{T_m} \int_{-F_m/2}^{F_m/2} h(\tau',\nu') e^{j2\pi\nu'(\tau-\tau')} \sqrt{T} \sum_{n=-\infty}^{\infty} s(\tau - \tau' + nT) e^{-j2\pi(\nu-\nu')nT} d\tau' d\nu'$$

$$\mathcal{Z}_r(\tau,\nu) = \int_0^{T_m} \int_{-F_m/2}^{F_m/2} h(\tau',\nu') \mathcal{Z}_s(\tau-\tau',\nu-\nu') e^{j2\pi\nu'(\tau-\tau')} d\tau' d\nu' = h(\tau,\nu) *_\sigma \mathcal{Z}_s(\tau,\nu)$$

→ **Twisted Convolution**

- **Zak domain relationship between input and output of an LTV channel.**

# Discrete Zak Transform and Discrete LTV Channels

- **The DZT of an $N$-length signal $x[n], n = 0, \ldots, N-1$ is given by***

$$Z_x[k,l] = \frac{1}{\sqrt{L}} \sum_{q=0}^{L-1} x[k+qK] e^{-j2\pi \frac{lq}{L}}$$

**For a fixed $k$** ➡ $x[k+qK] = x_k[q]$

**Collection of DFTs** ➡ $= \frac{1}{\sqrt{L}} \sum_{q=0}^{L-1} x_k[q] e^{-j2\pi \frac{lq}{L}}$

  - **Delay $k = 0, \ldots, K-1$ and Doppler $l = 0, \ldots, L-1$**
  - **Two parameters $K$ and $L$ which must satisfy $KL = N$**

- **DZT basis functions:**

$$p_{k,l}[n] \triangleq \frac{1}{\sqrt{L}} \sum_{q=0}^{L-1} \delta[n-k-qK] e^{j2\pi(l/L)(q)}$$

- **DZT analysis and synthesis equations:**

$$Z_x[k,l] = \sum_{n=0}^{N-1} x[n] p^*_{k,l}[n]$$

$$x[n] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} Z_x[k,l] p_{k,l}[n]$$



[1] H. Bolcskei and F. Hlawatsch, "Discrete Zak transforms, polyphase transforms, and applications," *IEEE Trans on SP*, Apr. 1997.

# Discrete LTV Channel and DZT

- **Consider the discrete LTV channel given by**

$$r[n] = \sum_{k=0}^{K_c-1} \sum_{l=0}^{L_c-1} h[k,l] e^{j2\pi(l/N)(n-k)} s[n-k]$$

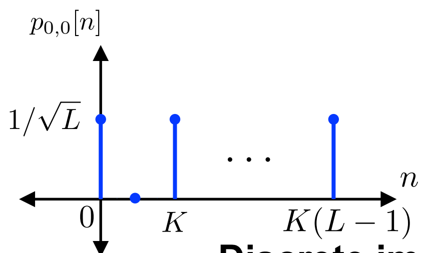
$K_c$ **and** $L_c$ **: Max. delay and Doppler taps**

- **For the input** $s[n] = p_{0,0}[n] = \frac{1}{\sqrt{L}} \sum_{q=0}^{L-1} \delta[n-qK]$**, the output** $u[n]$ **is given by:**

$$u[n] = \sum_{k=0}^{K_c-1} \sum_{l=0}^{L_c-1} h[k,l] p_{k,l}[n]$$

**DZT synthesis**

$$x[n] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} Z_x[k,l] p_{k,l}[n]$$




**Discrete impulse train**

$p_{0,0}[n] \longrightarrow$ **Underspread LTV Channel** $\longrightarrow u[n] \overset{\textbf{DZT}}{\longleftrightarrow} h[k,l]$

**Discrete impulse train response** — **Discrete Zak response**

$$Z_s[k,l] \longleftrightarrow s[n] \longrightarrow \boxed{h[k,l]} \longrightarrow r[n] \longleftrightarrow Z_r[k,l] = \sum_{l'=0}^{L_c-1} \sum_{k'=0}^{K_c-1} h[k',l'] Z_s[k-k', l-l'] e^{j2\pi\frac{l'(k-k')}{N}}$$

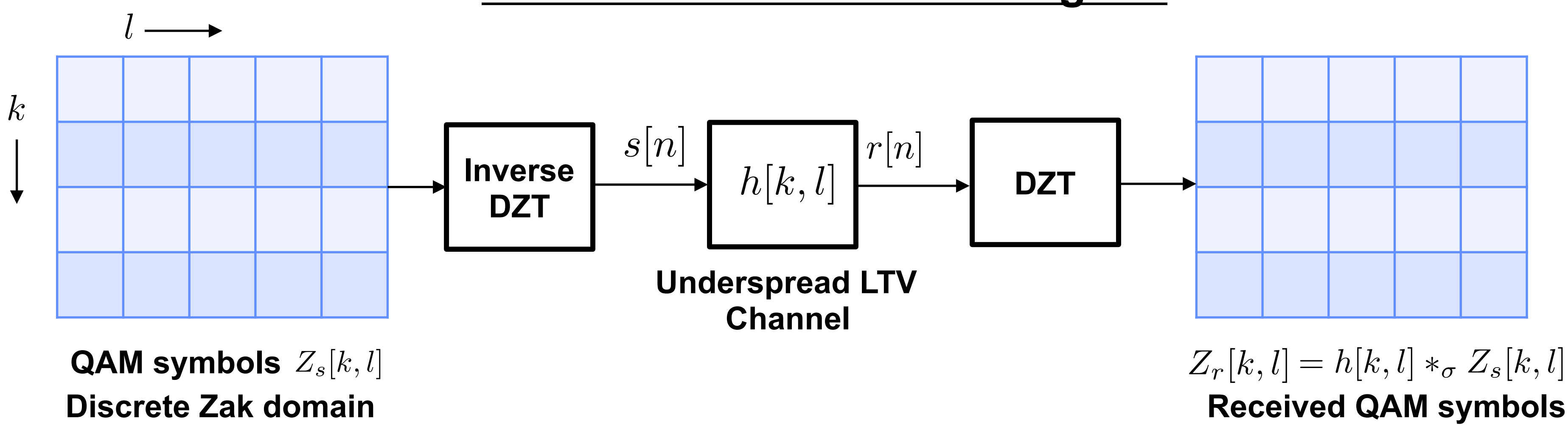
**(Discrete Twisted Convolution)**

# DZT and OTFS Modulation

- **Discrete twisted convolution relation for LTV channel:**

$$Z_r[k,l] = \sum_{l'=0}^{L_c-1} \sum_{k'=0}^{K_c-1} h[k',l'] Z_s[k-k',l-l'] e^{j2\pi \frac{l'(k-k')}{N}} = h[k,l] *_\sigma Z_s[k,l]$$

## DZT-based OTFS block diagram



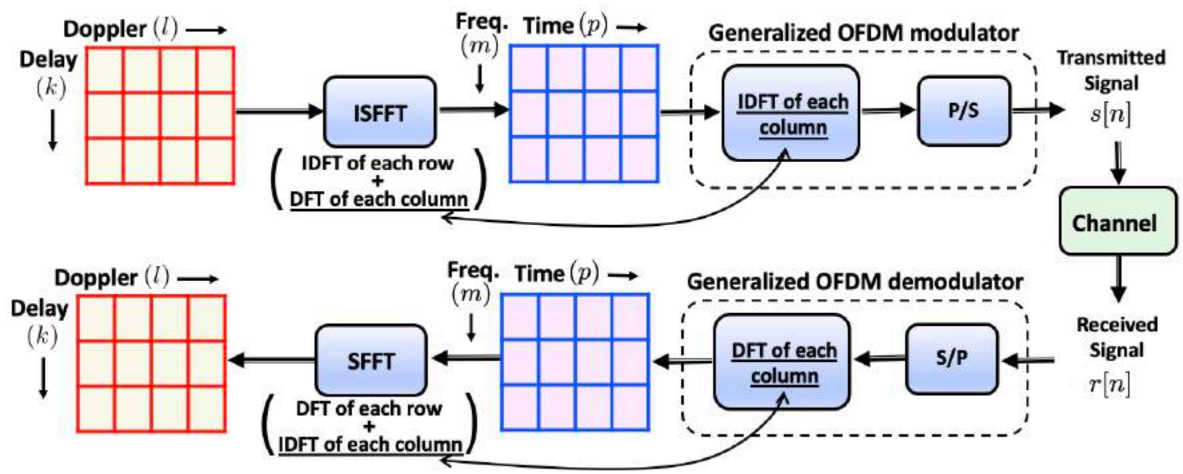

  - A suitable equalizer required to recover transmitted symbols

# Connections to OFDM-based Implementation of OTFS

**OTFS as an overlay over existing OFDM system** ➡

Transmitter

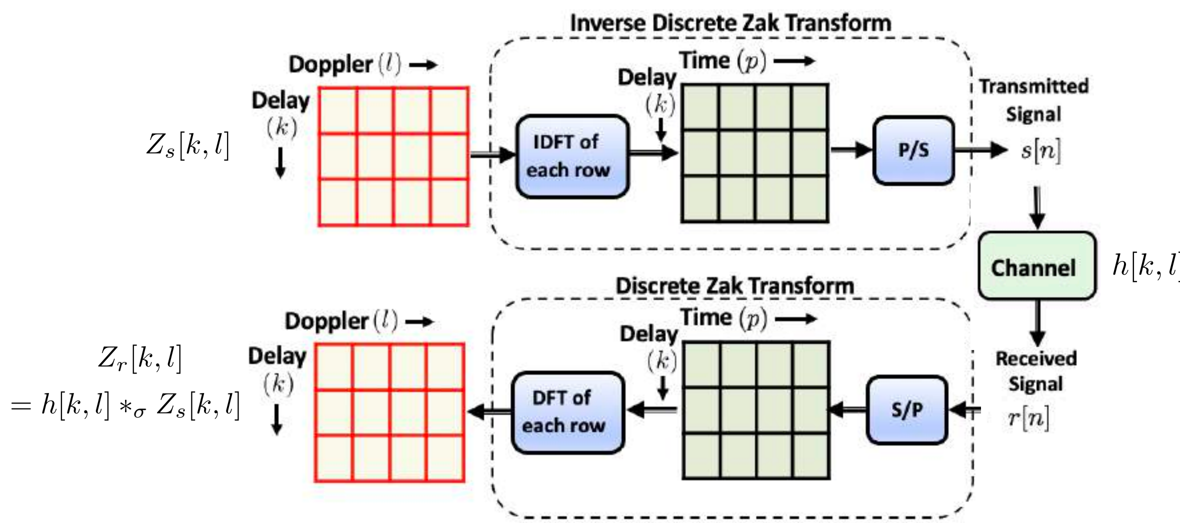
Receiver




**DZT-based interpretation of OTFS** ➡

Transmitter

Receiver



# Outline

- Wireless Communications Channel Models
- Orthogonal Frequency Division Multiplexing (OFDM)
- The Zak Transform
- OTFS: Modulation based on the Zak Transform
- **Summary**

# Summary

- **The spreading function $h(\tau, \nu)$ is the Zak response for an underspread LTV channel**
- **Derived the Zak domain channel equation for LTV channels**
- **Derived DZT analysis and synthesis equations**
- **Derived the discrete Zak domain channel equation for LTV channels**
- **OTFS: Modulation scheme based on DZT twisted convolution relation**
- **DZT-based interpretation can lend insight into OTFS analysis**

# Outline

- Introduction
- Akshay discussion
- **Closing remarks**

# Closing Remarks / Discussion

- **Zak Transform is natural tool for analysis of OTFS modulation as well as multipath channels**
  - **Study fundamental tradeoff between mitigating delay-Doppler spread and improving spectral efficiency of OTFS transmission.**
  - **Study how OTFS impulse response defines a shared secret that would enable physical layer security.**
    - **Consider MIMO extensions**
- **Can leverage neural network-based methods for OTFS equalization**

Thank You